

Optimal dividend problems for a jump-diffusion model with capital injections and proportional transaction costs

^aCHUANCUN YIN and ^bKAM CHUEN YUEN

^a *School of Statistics, Qufu Normal University,
Shandong 273165, P.R. China*

Corresponding author: E-mail: ccyin@mail.qfnu.edu.cn

^b *Department of Statistics and Actuarial Science, The University of Hong Kong,
Pokfulam Road, Hong Kong
E-mail: kcyuen@hku.hk*

September 2, 2014

Abstract In this paper, we study the optimal control problem for a company whose surplus process evolves as an upward jump diffusion with random return on investment. Three types of practical optimization problems faced by a company that can control its liquid reserves by paying dividends and injecting capital. In the first problem, we consider the classical dividend problem without capital injections. The second problem aims at maximizing the expected discounted dividend payments minus the expected discounted costs of capital injections over strategies with positive surplus at all times. The third problem has the same objective as the second one, but without the constraints on capital injections. Under the assumption of proportional transaction costs, we identify the value function and the optimal strategies for any distribution of gains.

Key words and phrases. Barrier strategy, dual model, HJB equation, jump-diffusion,

optimal dividend strategy, stochastic control.

Mathematics Subject Classification (2000). Primary: 93E20, 91G80 Secondary: 60J75.

1 INTRODUCTION

For the optimal dividend problem, one may adopt the objective of maximizing the expectation of the discounted dividends until possible ruin. This problem was first addressed by De Finetti [16] who considered a discrete time risk model with step sizes ± 1 and showed that the optimal dividend strategy is a barrier strategy. Miyasawa [21] generalized the model to the case that periodic gains of a company can take on values $-1, 0, 1, 2, 3, \dots$, and showed that the optimal dividend strategy of the generalized model is a barrier one. Subsequently, the problem of finding the optimal dividend strategy has attracted great attention in the literature of insurance mathematics. For nice surveys on this topic, we refer the reader to Avanzi [3] and Schmidli [22]. Besides insurance risk models, the optimal dividend problem in the so-called dual model has also been studied extensively in recent years. Among others, Avanzi et al. [6] discussed how the expectation of the discounted dividends until ruin can be calculated for the dual model when the gain amounts follow an exponential distribution or a mixture of exponential distributions, and showed how the exact value of the optimal dividend barrier can be determined; and Avanzi and Gerber [5] examined the same problem for the dual model that is perturbed by diffusion, and showed that the optimal dividend strategy in the dual model is also a barrier strategy. To make the problem more interesting, the issue of capital injections has also been considered in the study of optimal dividends in the dual model. Yao et al. [23] studied the optimal problem with dividend payments and issuance of equity in the dual model with proportional transaction costs, and derived the optimal strategy that maximizes the expected present value of dividend payments minus the discounted costs of issuing new equity before ruin. Yao et al. [24] considered the same problem with both fixed and proportional transaction costs. Dai et al. [14,15] investigated the same problem as in Yao et al. [23] for the dual model with diffusion with bounded gains and exponential gains,

respectively. Avanzi et al. [7] derived an explicit expression for the value function in the dual model with diffusion when the gains distribution is a mixture of exponentials in the presence of both dividends and capital injections. Specifically, they showed that barrier dividend strategy is optimal, and conjectured that the optimal dividend strategy in the dual model with diffusion should be the barrier strategy regardless of the distribution of gains. Bayraktar et al. [11] examined the same cash injection problem, and used the fluctuation theory of spectrally positive Lévy processes to show the optimality of the barrier strategy for all positive Lévy processes. Bayraktar et al. [12] extended the study to the case with fixed transaction costs. Other related work can be found in Yin and Wen [26], Yin, Wen and Zhao [28], Avanzi et al. [8], Yao et al. [25] and Zhang [29].

In this paper, we provide a uniform mathematical framework to analyze the optimal control problem with dividends and capital injections in the presence of proportional transaction costs for the dual model with random return on investment. The associated value function is defined as the expected present value of dividends minus costs of capital injections until ruin. The rest of the paper is organized as follows. In Section 2, we give a rigorous mathematical formulation of the problem. Section 3 works on the model without capital injections, while Section 4 deals with the model with capital injections which never goes bankrupt. Finally, we solve the general stochastic control problem in Section 5.

2 Problem formulation

Assume that the surplus generating process P_t at time t is given by

$$P_t = x - pt + \sigma_p W_{p,t} + \sum_{i=1}^{N_t} X_i, \quad t \geq 0, \quad (2.1)$$

where $x > 0$ is the initial assets, p and σ_p are positive constants, $\{W_{p,t}\}_{t \geq 0}$ is a standard Brownian motion independent of the homogeneous compound Poisson process $\sum_{i=1}^{N_t} X_i$, and $\{X_i\}$ is a sequence of independent and identically distributed random variables having common distribution function F with $F(0) = 0$. Let λ be the intensity of the Poisson process N_t . We assume throughout the paper that $E[X_i] < \infty$ and $\lambda E[X_i] - p > 0$. Here,

we consider the return on investment generating process

$$R_t = rt + \sigma_R W_{R,t}, \quad t \geq 0, \quad (2.2)$$

where $\{W_{R,t}\}_{t \geq 0}$ is another standard Brownian motion, and r and σ_R are positive constants. It is assumed that $W_{p,t}$ and $W_{R,t}$ are correlated in the way that

$$W_{R,t} = \rho W_{p,t} + \sqrt{1 - \rho^2} W_{p,t}^0,$$

where $\rho \in [-1, 1]$ is constant, and $W_{p,t}^0$ is a standard Brownian motion independent of $W_{p,t}$.

Define the risk process U_t as the total assets of the company at time t , i.e., U_t is the solution to the stochastic differential equation

$$U_t = P_t + \int_0^t U_{s-} dR_s, \quad t \geq 0. \quad (2.3)$$

The solution to (2.3) is given by (see, e.g. Jaschke [19, Theorem 1])

$$U_t = \mathcal{E}(R)_t \left(x + \int_0^t \mathcal{E}(R)_{s-}^{-1} dP_s - \rho \sigma_p \sigma_R \int_0^t \mathcal{E}(R)_{s-}^{-1} ds \right),$$

where

$$\mathcal{E}(R)_t = \exp\left\{\left(r - \frac{1}{2}\sigma_R^2\right)t + \sigma_R W_{R,t}\right\}.$$

Using Itô's formula for semimartingale, one can show that the infinitesimal generator \mathcal{L} of $U = \{U_t, t \geq 0\}$ is given by

$$\begin{aligned} \mathcal{L}g(y) = & (ry - p)g'(y) + \frac{1}{2} [(\sigma_p + \rho\sigma_R y)^2 + \sigma_R^2(1 - \rho^2)y^2] g''(y) \\ & + \lambda \int_0^\infty [g(y+z) - g(y)] F(dz). \end{aligned} \quad (2.4)$$

The model (2.3) is a natural extension of the dual model in Avanzi and Gerber [5] and Avanzi et al. [6]. As was mentioned in Avanzi et al. [6], the dual model is appropriate for companies that have deterministic expenses and occasional gains whose amount and frequency can be modelled by the jump process $\sum_{i=1}^{N_t} X_i$. For example, for companies such as pharmaceutical or petroleum companies, the jump could be interpreted as the net present value of future gains from an invention or discovery. Another example is the

venture capital investments or research and development investments. Venture capital funds screen out start-up companies and select some companies to invest in. When there is a technological breakthrough, the jump is generated. More examples can be found in Bayraktar and Egami [10] and Avanzi and Gerber [5].

In this paper, we denote by L_t the cumulative amount of dividends paid up to time t with $L_{0-} = 0$, and by G_t the total amount of capital injections up to time t with $G_{0-} = 0$. A dividend control strategy ξ is described by the stochastic process $\xi = (L_t, G_t)$. A strategy is called admissible if both L and G are non-decreasing $\{\mathcal{F}_t\}$ -adapted processes, and their sample paths are right-continuous with left limits. We denote by Ξ the set of all admissible dividend policies. The risk process with initial capital $x \geq 0$ and controlled by a strategy ξ is given by $U^\xi = \{U_t^\xi, t \geq 0\}$, where U_t^ξ is the solution to the stochastic differential equation

$$dU_t^\xi = dP_t + U_{t-}^\xi dR_t - dL_t^\xi + dG_t^\xi, \quad t \geq 0.$$

Moreover, $L_t^\xi - L_{t-}^\xi \leq U_{t-}^\xi$ for all t . In words, the amount of dividends is smaller than the size of the available capitals. Let $\tau^\xi = \inf\{t \geq 0 : U_t^\xi = 0\}$ be the ruin time. Then, the associated performance function is given by

$$V(x; \xi) = E_x \left(\alpha \int_{0-}^{\tau^\xi-} e^{-\delta t} dL_t^\xi - \beta \int_{0-}^{\tau^\xi-} e^{-\delta t} dG_t^\xi \right), \quad (2.5)$$

where $\delta > 0$ is the discounted rate, $1 - \alpha$ ($0 < \alpha \leq 1$) is the rate of proportional costs on dividend transactions, $1 \leq \beta < \infty$ is the rate of proportional transaction costs of capital injections. The notation E_x represents the expectation conditioned on $U_0^\xi = x$ and the integral is understood pathwise in a Lebesgue-Stieltjes sense. Our aim is to find the value function

$$V_*(x) = \sup_{\xi \in \Xi} V(x; \xi), \quad (2.6)$$

and the optimal policy $\xi^* \in \Xi$ such that $V(x; \xi^*) = V_*(x)$ for all $x \geq 0$.

The study of optimal dividends has been around many years. The commonly-used approach to solving these optimal control problems is to proceed by guessing a candidate optimal solution, constructing the corresponding value function, and subsequently

verifying its optimality through a verification result. For the model of study, i.e., an upward jump-diffusion process with random return on investment, the optimal control problem remains to be solved. The problem of study can be seen as a natural extension of Bayraktar and Egami [10], and Avanzi, Shen and Wong [7]. In addition, one can see later that the method used in Bayraktar, Kyprianou and Yamazaki [11] cannot be applied to our model since their proof relies on certain characteristics of Lévy process. In order to solve the optimal control problem in this paper, we shall first consider two sub-optimal problems in the next two sections.

3 Optimal dividend problem without capital injections

In this section, we first consider the dividend problem without capital injections. We shall show that the barrier strategy solve the optimal dividend problem regardless of the jump distribution.

Let $\Xi_d = \{\xi_d = (L^{\xi_d}, G^{\xi_d}) : (L^{\xi_d}, G^{\xi_d}) \in \Xi \text{ and } G^{\xi_d} \equiv 0\}$. The associated controlled process is denoted by $U^{\xi_d} = \{U_t^{\xi_d}, t \geq 0\}$, where $U_t^{\xi_d}$ is the solution to the stochastic differential equation

$$dU_t^{\xi_d} = dP_t + U_t^{\xi_d} dR_t - dL_t^{\xi_d}, \quad t \geq 0.$$

and the value function is given by

$$V_d(x) = \sup_{\xi_d \in \Xi_d} V(x; \xi_d) \equiv \sup_{\xi_d \in \Xi_d} E_x \left(\alpha \int_{0-}^{\tau_{\xi_d}^-} e^{-\delta t} dL_t^{\xi_d} \right), \quad x \geq 0, \quad (3.1)$$

where $\tau_{\xi_d} = \inf\{t : U_t^{\xi_d} = 0\}$ is the time of ruin under the strategy ξ_d . We next identify the form of the value function V_d and the optimal strategy ξ_d^* such that $V_d(x) = V(x; \xi_d^*)$.

3.1 HJB equation and verification lemma

For notational convenience, denote $v(x) = V(x; \xi_d^*)$. If v is twice continuously differentiable, then applying standard arguments from stochastic control theory (see Fleming and Soner [17]) or an approach similar to that in Azcue and Muler [9], we can show that

the value function fulfils the dynamic programming principle

$$v(x) = \sup_{\xi_d \in \Xi} E_x \left(\int_0^{\tau_{\xi_d} \wedge T} e^{-\delta s} dL_s^{\xi_d} + e^{-\delta(\tau_{\xi_d} \wedge T)} v(U_{\tau_{\xi_d} \wedge T}^{\xi_d}) \right),$$

for any stopping time T , and that the associated Hamilton-Jacobi-Bellman (HJB) equation is

$$\max\{\mathcal{L}v(x) - \delta v(x), \alpha - v'(x)\} = 0, \quad x > 0, \quad (3.2)$$

with $v(0) = 0$, where \mathcal{L} is the extended generator of U defined in (2.4). The HJB equation (3.2) can also be obtained by the heuristic argument of Avanzi et al. [7].

Lemma 3.1. (*Verification Lemma*) *Let v be a solution to (3.2). Then, $v(x) \geq V(x; \xi_d)$ for any admissible strategy $\xi_d \in \Xi_d$, and thus $v(x) \geq V_d(x)$.*

Proof. For any admissible strategy $\xi_d \in \Xi_d$, put $\Lambda = \{s : L_{s-}^{\xi_d} \neq L_s^{\xi_d}\}$. Applying Ito's formula for semimartingale to $e^{-\delta t} v(U_t^{\xi_d})$ gives

$$\begin{aligned} E_x[e^{-\delta(t \wedge \tau_{\xi_d}^-)} v(U_{t \wedge \tau_{\xi_d}^-}^{\xi_d})] &= v(x) + E_x \int_0^{t \wedge \tau_{\xi_d}^-} e^{-\delta s} (\mathcal{L} - \delta) v(U_{s-}^{\xi_d}) ds \\ &\quad + E_x \sum_{s \in \Lambda, s \leq t \wedge \tau_{\xi_d}^-} e^{-\delta s} \left\{ v(U_s^{\xi_d}) - v(U_{s-}^{\xi_d}) \right\} \\ &\quad - E_x \int_{0-}^{t \wedge \tau_{\xi_d}^-} e^{-\delta s} v'(U_{s-}^{\xi_d}) dL_s^{\xi_d, c}, \end{aligned} \quad (3.3)$$

where $L_s^{\xi_d, c}$ is the continuous part of $L_s^{\xi_d}$. From (3.2), we see that $(\mathcal{L} - \delta)v(U_{s-}^{\xi_d}) \leq 0$ and $v'(x) \geq \alpha$. Thus, for $s \in \Lambda, s \leq t \wedge \tau_{\xi_d}$,

$$v(U_s^{\xi_d}) - v(U_{s-}^{\xi_d}) \leq -\alpha(L_s^{\xi_d} - L_{s-}^{\xi_d}). \quad (3.4)$$

It follows from (3.3) and (3.4) that

$$E_x[e^{-\delta(t \wedge \tau_{\xi_d}^-)} v(U_{t \wedge \tau_{\xi_d}^-}^{\xi_d})] \leq v(x) - \alpha E_x \int_{0-}^{t \wedge \tau_{\xi_d}^-} e^{-\delta s} dL_s^{\xi_d}. \quad (3.5)$$

Letting $t \rightarrow \infty$ in (3.5) yields the result. \square

3.2 Construction of a candidate solution

It is assumed that dividends are paid according to the barrier strategy ξ_b . Such a strategy has a level of barrier $b > 0$. When the surplus exceeds the barrier, the excess is

paid out immediately as dividends. Let L_t^b be the total amount of dividends up to time t . The controlled risk process when taking into account of the dividend strategy ξ_b is $U^b = \{U_t^b, t \geq 0\}$, where U_t^b is the solution to the following stochastic differential equation

$$dU_t^b = dP_t + U_{t-}^b dR_t - dL_t^b, \quad t \geq 0.$$

Denote by $V_b(x)$ the expected discounted dividends function if the barrier strategy ξ_b is applied, that is,

$$V_b(x) = \alpha E_x \left(\int_{0-}^{T_b^x-} e^{-\delta t} dL_t^b \right), \quad (3.6)$$

where $\delta > 0$ is the force of interest and $T_b^x = \inf\{t \geq 0 : U_t^b = 0\}$.

The following result shows that $V_b(x)$ as a function of x satisfies an integro-differential equation with certain boundary conditions.

Lemma 3.2. *For the risk process U of (2.3) and the infinitesimal generator \mathcal{L} of (2.4), if $h_b(x)$ solves*

$$\mathcal{L}h_b(x) = \delta h_b(x), \quad 0 < x < b,$$

and $h_b(x) = h_b(b) + \alpha(x - b)$, for $x > b$, together with the boundary conditions

$$h_b(0) = 0, \quad h'_b(b) = \alpha,$$

then $h_b(x)$ coincides with $V_b(x)$ given by (3.6).

Proof. Applying Ito's formula for semimartingale to $e^{-\delta t} h_b(U_{t-}^b)$ gives

$$\begin{aligned} e^{-\delta t} h_b(U_{t-}^b) - h_b(U_0^b) &= \int_{0-}^{t-} e^{-\delta s} dN_s^b + \int_0^t e^{-\delta s} (\mathcal{L} - \delta) h_b(U_{s-}^b) ds \\ &+ \sum_{s \leq t} \mathbf{1}_{\{\Delta L_s > 0\}} e^{-\delta s} \{h_b(U_{s-}^b + \Delta P_s - \Delta L_s) - h_b(U_{s-}^b + \Delta P_s)\} \\ &- \int_{0-}^{t-} e^{-\delta s} h'_b(U_{s-}^b) dL_s^c, \end{aligned} \quad (3.7)$$

where L_s^c is the continuous part of L_s , and

$$\begin{aligned} N_t^b &= \sum_{s \leq t} \mathbf{1}_{\{|\Delta P_s| > 0\}} \{h_b(U_{s-}^b + \Delta P_s) - h_b(U_{s-}^b)\} \\ &- \int_0^t \int_0^\infty \{h_b(U_{s-}^b + y) - h_b(U_{s-}^b)\} \Pi(dy) ds \\ &+ \sigma \int_0^t h'_b(U_{s-}^b) dW_s. \end{aligned}$$

Note that $P(\Delta L_s > 0, \Delta P_s < 0) = 0$ and that $U_{s-}^b + \Delta P_s \geq U_{s-}^b + \Delta P_s - \Delta L_s \geq b$ on $\{\Delta L_s > 0, \Delta P_s > 0\}$. Consequently,

$$\begin{aligned} \sum_{s < t} \mathbf{1}_{\{\Delta L_s > 0\}} e^{-\delta s} \{h_b(U_{s-}^b + \Delta P_s - \Delta L_s) - h_b(U_{s-}^b + \Delta P_s)\} \\ = -\alpha \sum_{s < t} \mathbf{1}_{\{\Delta L_s > 0\}} e^{-\delta s} \Delta L_s. \end{aligned}$$

Note that N_t^b is a local martingale, and

$$\int_{0-}^{t-} e^{-\delta s} h'_b(U_{s-}^b) dL_s^c = \int_{0-}^{t-} e^{-\delta s} h'_b(U_s^b) dL_s^c = \alpha \int_{0-}^{t-} e^{-\delta s} h'_b(b) dL_s^c.$$

Thus, for any appropriate localization sequence of stopping times $\{t_n, n \geq 1\}$, we have

$$E_x(e^{-\delta(t_n \wedge T^b)} h_b(U_{t_n \wedge T^b}^b)) - E_x h_b(U_0^b) = -\alpha E_x \int_{0-}^{t_n \wedge T^b-} e^{-\delta s} dL_s. \quad (3.8)$$

Letting $n \rightarrow \infty$ in (3.8) yields the result. \square

Lemma 3.3. $V_b(x)$ is a concave increasing function on $(0, \infty)$.

Proof. To prove the lemma, we use arguments similar to those in Kulenko and Schmidli [20]. Let $x > 0, y > 0$, and $l \in (0, 1)$. Consider the strategies L^x and L^y for the initial capitals x and y . Define $L_t = lL_t^x + (1-l)L_t^y$. Then, $L_t = L_t^{lx+(1-l)y}$. Since the processes $\{P_t, t \geq 0\}$ and $\{R_t, t \geq 0\}$ have no negative jumps, we have $\tau_L = \tau_{L^x} \vee \tau_{L^y}$. It follows that

$$\begin{aligned} V_b(lx + (1-l)y) &= \alpha E_x \left(\int_{0-}^{\tau_L-} e^{-\delta t} dL_t \right) \\ &= \alpha l E_x \left(\int_{0-}^{\tau_L-} e^{-\delta t} dL_t^x \right) + \alpha(1-l) E_x \left(\int_{0-}^{\tau_L-} e^{-\delta t} dL_t^y \right) \\ &\geq \alpha l E_x \left(\int_{0-}^{\tau_{L^x}-} e^{-\delta t} dL_t^x \right) + \alpha(1-l) E_x \left(\int_{0-}^{\tau_{L^y}-} e^{-\delta t} dL_t^y \right) \\ &= lV_b(x) + (1-l)V_b(y), \end{aligned}$$

and thus the concavity of V_b follows. The increasingness of $V_b(x)$ is trivial \square

3.3 Verification of optimality

Define the barrier level by

$$b^* = \sup\{b \geq 0 : V_b'(b-) = \alpha\}.$$

We conjecture that the barrier strategy ξ_{b^*} is optimal.

Proposition 3.1. $b^* = 0$ if and only if $\lambda \int_0^\infty yF(dy) \leq p$.

Proof. Here, we follow the approach of Yao et al. [23] to prove the proposition. Suppose that $b^* = 0$. Then, the associated value function is $V_d(x) = \alpha x$ which satisfies the HJB equation (3.2). As a result, we obtain $(\Gamma - \delta)V_d(x) \leq 0$ which in turn gives $\lambda \int_0^\infty yF(dy) \leq p$. On the other hand, suppose that $\lambda \int_0^\infty yF(dy) \leq p$. Then, $w(x) = \alpha x$ satisfies (3.2). By Lemma 3.1, we get $w(x) \geq V_d(x)$. However, $w(x) \leq V_d(x)$ since $w(x) = \alpha x$ is the performance function associated with the strategy that x is paid immediately as dividends. In this case, ruin occurs immediately. Thus, $w(x) = V_d(x)$ and the optimal barrier level $b^* = 0$. \square

Theorem 3.1. If $\lambda \int_0^\infty yF(dy) > p$, then the function V_{b^*} defined in (3.6) satisfies

$$V_{b^*}(x) = V_d(x), \quad x \geq 0,$$

and the optimal barrier strategy ξ_d^* is the solution to

$$dU_t^{\xi_d^*} = dP_t + U_{t-}^{\xi_d^*} dR_t - dL_t^{\xi_d^*}, \quad t \geq 0,$$

with the conditions

$$U_t^{\xi_d^*} \leq b^*, \quad G_t^{\xi_d^*} \equiv 0, \quad \int_0^\infty \mathbf{1}_{\{U_s^{\xi_d^*} < b^*\}} dL_s^{\xi_d^*} = 0.$$

Proof. Using the method of Avanzi and Gerber [5], it can be shown that $V_{b^*}(x)$ is twice continuously differentiable at $x = b^*$. Consequently, $V_{b^*} \in C^2(\mathbb{R}_+)$. Note that $(\mathcal{L} - \delta)V_{b^*}(x) = 0$ and $V_{b^*}'(x) \geq \alpha$ for $x \in [0, b^*)$ due to the concavity of V_{b^*} on $[0, b^*)$. Since $V_{b^*}(x) = \alpha(x - b^*) + V_{b^*}(b^*)$ for $x \geq b^*$, we have

$$\begin{aligned} (\mathcal{L} - \delta)V_{b^*}(x) &= -p\alpha + \alpha \int_0^\infty yF(dy) - \alpha(x - b^*) - \delta V_{b^*}(b^*) \\ &< -p\alpha + \alpha \int_0^\infty yF(dy) - \delta V_{b^*}(b^*) \\ &= \lim_{x \rightarrow b^*+} (\mathcal{L} - \delta)V_{b^*}(x) = \lim_{x \rightarrow b^*-} (\mathcal{L} - \delta)V_{b^*}(x) = 0, \end{aligned}$$

because of the continuity of V_{b^*} , V_{b^*}' , and V_{b^*}'' at $x = b^*$. Thus, the function V_{b^*} satisfies the HJB equation (3.2). Then, it follows from Lemma 3.1 that $V_{b^*}(x) \geq V_d(x)$. However, $V_{b^*}(x) \leq V_d(x)$ by definition, and hence $V_{b^*}(x) = V_d(x)$. \square

3.4 Two closed-form solutions

Owing to the complexity of the equation, the solution may not be available in explicit form in general. The following two examples show that one can derive closed-form solution in some special cases.

Example 3.1. Assume that $r = 0$ and $\sigma_R = 0$. Then, $V_{b^*}(x)$ satisfies the following integro-differential equation

$$\mathcal{A}V_{b^*}(x) = \delta V_{b^*}(x), \quad 0 < x < b^*, \quad (3.9)$$

and

$$V_{b^*}(x) = \alpha(x - b^*) + V_{b^*}(b^*), \quad x > b^*, \quad (3.10)$$

with the boundary conditions

$$V_{b^*}(0) = 0, \quad V_{b^*}'(x)|_{x=b^*} = \alpha, \quad (3.11)$$

where

$$\mathcal{A}g(x) = \frac{1}{2}\sigma_p^2 g''(x) - pg'(x) - \lambda g(x) + \lambda \int_0^\infty g(x+y)F(dy).$$

Following the arguments of Laplace transform used in Yin, Wen and Zhao [28], one can show that the solution to (3.9)-(3.11) is given by

$$V_{b^*}(x) = -\alpha \bar{Z}^{(\delta)}(b^* - x) + \alpha \frac{E[X_1]}{\delta},$$

and

$$b^* = (\bar{Z}^{(\delta)})^{-1} \left(\frac{E[X_1]}{\delta} \right),$$

where

$$Z^{(\delta)}(x) = 1 + \delta \int_0^x W^{(\delta)}(y)dy, \quad \bar{Z}^{(\delta)}(x) = \int_0^x Z^{(\delta)}(y)dy, \quad x \in \mathbb{R}.$$

Here, $W^{(\delta)}$ is the so-called δ -scale function defined in the way that $W^{(\delta)}(x) = 0$ for all $x < 0$ and that its Laplace transform on $[0, \infty)$ is given by

$$\int_0^\infty e^{-\theta x} W^{(\delta)}(x)dx = \frac{1}{\Psi(\theta) - \delta}, \quad \theta > \sup\{\theta \geq 0 : \Psi(\theta) = \delta\},$$

where

$$\Psi(\theta) = p\theta + \frac{1}{2}\sigma_p^2\theta^2 + \lambda \int_0^\infty (e^{-\theta x} - 1)F(dx).$$

For further details, the reader is referred to Yin and Wen [26]. \square

Example 3.2. Let $\sigma_R = \sigma_p = 0$. Assume that X_i is exponentially distributed with parameter μ . Then, by Theorem 3.1 and Lemma 3.2, it can be shown that $V_{b^*}(x)$ satisfies the following integro-differential equation

$$(rx - p)V_{b^*}'(x) + \lambda\mu \int_0^\infty V_{b^*}(x + z)e^{-\mu z}dz = (\lambda + \delta)V_{b^*}(x), \quad 0 < x < b^*, \quad (3.12)$$

and

$$V_{b^*}(x) = \alpha(x - b^*) + V_{b^*}(b^*), \quad x > b^*, \quad (3.13)$$

with the boundary conditions

$$V_{b^*}(0) = 0, \quad V_{b^*}'(x)|_{x=b^*} = \alpha. \quad (3.14)$$

From equation (3.12), we find that

$$zg''(z) + \left(1 - \frac{\lambda + \delta}{r} - z\right)g'(z) + \frac{\delta}{r}g(z) = 0,$$

where

$$g(z) = V_{b^*}(x), \quad z = \mu \left(x - \frac{p}{r}\right).$$

Note that this is Kummer's confluent hypergeometric equation with the solution given by

$$g(z) = C_1 M\left(-\frac{\delta}{r}, 1 - \frac{\lambda + \delta}{r}, z\right) + C_2 U\left(-\frac{\delta}{r}, 1 - \frac{\lambda + \delta}{r}, z\right),$$

where C_1 and C_2 are constants, and $M(a, b, x)$ is the standard confluent hypergeometric function with $U(a, b, x)$ being its second form; see, for example, Abramowitz and Stugen [1, pp. 504-505]. Then, it follows that

$$V_{b^*}(x) = C_1 M\left(-\frac{\delta}{r}, 1 - \frac{\lambda + \delta}{r}, \mu(x - \frac{p}{r})\right) + C_2 U\left(-\frac{\delta}{r}, 1 - \frac{\lambda + \delta}{r}, \mu(x - \frac{p}{r})\right).$$

Using the boundary conditions (3.14) and the formulae

$$M'(a, b, z) = \frac{a}{b}M(a + 1, b + 1, z), \quad U'(a, b, z) = -aU(a + 1, b + 1, z),$$

we obtain the coefficients

$$C_1 = \frac{\alpha U(-\frac{\delta}{r}, 1 - \frac{\lambda+\delta}{r}, -\frac{\mu p}{r})}{\Delta(b^*)},$$

and

$$C_2 = -\frac{\alpha M(-\frac{\delta}{r}, 1 - \frac{\lambda+\delta}{r}, -\frac{\mu p}{r})}{\Delta(b^*)},$$

where

$$\begin{aligned} \Delta(b^*) &= -\frac{\mu\delta}{r-\lambda-\delta} U\left(-\frac{\delta}{r}, 1 - \frac{\lambda+\delta}{r}, -\frac{\mu p}{r}\right) M\left(1 - \frac{\delta}{r}, 2 - \frac{\lambda+\delta}{r}, \mu(b^* - \frac{p}{r})\right) \\ &\quad + \frac{\mu\delta}{r} M\left(-\frac{\delta}{r}, 1 - \frac{\lambda+\delta}{r}, -\frac{\mu p}{r}\right) U\left(1 - \frac{\delta}{r}, 2 - \frac{\lambda+\delta}{r}, \mu(b^* - \frac{p}{r})\right), \end{aligned}$$

and b^* is the maximizer of term $1/\Delta(b)$ with respect to b , i.e.,

$$b^* = \operatorname{argmax} \frac{1}{\Delta(b)}.$$

□

4 Optimal dividend problem with capital injections

In this section, we consider the optimal dividend problem with capital injections. The set of admissible strategies is given by

$$\Xi_c = \{\xi_c = (L^{\xi_c}, G^{\xi_c}) : (L^{\xi_c}, G^{\xi_c}) \in \Xi \text{ and } U_t^{\xi_c} \geq 0\}.$$

The controlled surplus process $U_t^{\xi_c}$ satisfies

$$dU_t^{\xi_c} = dP_t + U_{t-}^{\xi_c} dR_t - dL_t^{\xi_c} + dG_t^{\xi_c}, \quad t \geq 0,$$

and the value function is defined as

$$V_c(x) = \sup_{\xi_c \in \Xi_c} V(x; \xi_c) \equiv \sup_{\xi_c \in \Xi_c} E_x \left(\alpha \int_{0-}^{\infty} e^{-\delta t} dL_t^{\xi_c} - \beta \int_{0-}^{\infty} e^{-\delta t} dG_t^{\xi_c} \right), \quad x \geq 0. \quad (4.1)$$

Since the controlled surplus process always stays positive, the company will never go bankrupt. We shall identify the form of the value function V_c and the optimal strategy ξ_c^* such that $V_c(x) = V(x; \xi_c^*)$.

4.1 HJB equation and verification lemma

Applying the techniques used in Section 3, we get the HJB equation and the verification Lemma.

$$\max\{\mathcal{L}w(x) - \delta w(x), \alpha - w'(x), w'(x) - \beta\} = 0, \quad x \geq 0. \quad (4.2)$$

Lemma 4.1. (*Verification Lemma*) *Let w be a solution to (4.2). Then, $w(x) \geq V(x; \xi_c)$ for any admissible strategy $\xi_c \in \Xi_c$, and thus $w(x) \geq V_c(x)$.*

4.2 Construction of a candidate solution

We now construct a concave C^2 solution H to the HJB equation (4.2). Due to the effect of the discount factor, it is clear that the optimal strategy is the one that postpone capital injections as long as possible, i.e., we inject capital only when surplus become zero. Consider the barrier strategy with the upper barrier B^* and the lower barrier 0, and the strategy $\pi^* = (L^{\pi^*}, G^{\pi^*})$ where $(U_t^{\pi^*}, L_t^{\pi^*,x}, G_t^{\pi^*,x})$ is a solution to the following system

$$dU_t^{\pi^*} = dP_t + U_{t-}^{\pi^*} dR_t - dL_t^{\pi^*} + dG_t^{\pi^*}, \quad (4.3)$$

$$0 \leq U_t^{\pi^*} \leq B^*, \quad t \geq 0, \quad (4.4)$$

$$L_t^{\pi^*,x} = \max(x - B^*, 0) + \int_{0-}^{t-} 1(U_s^{\pi^*} = B^*) dL_s^{\pi^*}, \quad t > 0, \quad (4.5)$$

$$G_t^{\pi^*,x} = \max\left(-\inf_{0 \leq s \leq t} (P_s - L_s^{\pi^*}), 0\right), \quad t > 0. \quad (4.6)$$

Lemma 4.2. *For the problem of (4.3)-(4.6), if $H(x)$ solves*

$$\mathcal{L}H(x) = \delta H(x), \quad 0 < x < B^*,$$

with $H(x) = H(B^) + \alpha(x - B^*)$ for $x > B^*$ and the boundary conditions*

$$H'(0) = \beta, \quad H'(B^*) = \alpha,$$

where the infinitesimal generator \mathcal{L} is given by (2.4), then $H(x)$ is given by

$$H(x) = V(x; \pi^*) \equiv E_x \left(\alpha \int_{0-}^{\infty} e^{-\delta t} dL_t^{\pi^*,x} - \beta \int_{0-}^{\infty} e^{-\delta t} dG_t^{\pi^*,x} \right), \quad x \geq 0. \quad (4.7)$$

Proof. For the strategy π^* , define $\Lambda = \{s : L_{s-}^{\pi^*,x} \neq L_s^{\pi^*,x}\}$. Let $L_t^{\pi^*,x,c}$ be the continuous part of $L_t^{\pi^*,x}$. Since the process is skip-free downward, $G_t^{\pi^*,x}$ is continuous. In addition, we

see from (4.6) that $G_t^{\pi^*,x} \geq 0$ and that the support of the Stieltjes measure $dG_t^{\pi^*,x}$ is contained in the closure of the set $\{t : U_t^{\pi^*} = 0\}$. Applying Ito's formula for semimartingale to $e^{-\delta t} H(U_t^{\pi^*})$ gives

$$\begin{aligned} E_x[e^{-\delta t} H(U_{t-}^{\pi^*})] &= H(x) + E_x \int_0^t e^{-\delta s} (\mathcal{L} - \delta) H(U_s^{\pi^*}) ds \\ &\quad + E_x \sum_{s \in \Lambda, s \leq t} e^{-\delta s} \{H(U_s^{\pi^*}) - H(U_{s-}^{\pi^*})\} \\ &\quad - E_x \int_{0-}^{t-} e^{-\delta s} H'(U_{s-}^{\pi^*}) dL_s^{\pi^*,x,c} \\ &\quad + E_x \int_{0-}^{t-} e^{-\delta s} H'(U_{s-}^{\pi^*}) dG_s^{\pi^*,x}. \end{aligned} \quad (4.8)$$

Note that $(\mathcal{L} - \delta)H(U_s^{\pi^*}) = 0$, and that

$$\begin{aligned} E_x \sum_{s \in \Lambda, s \leq t} e^{-\delta s} \{H(U_s^{\pi^*}) - H(U_{s-}^{\pi^*})\} &= \alpha \sum_{s \leq t} e^{-\delta s} (L_s^{\pi^*,x} - L_{s-}^{\pi^*,x}), \\ E_x \int_{0-}^{t-} e^{-\delta s} H'(U_{s-}^{\pi^*}) dL_s^{\pi^*,x,c} &= E_x \int_{0-}^{t-} e^{-\delta s} H'(U_{s-}^{\pi^*}) dL_s^{\pi^*,x,c} = \alpha E_x \int_{0-}^{t-} e^{-\delta s} dL_s^{\pi^*,x,c}, \\ E_x \int_{0-}^{t-} e^{-\delta s} H'(U_{s-}^{\pi^*}) dG_s^{\pi^*,x} &= E_x \int_{0-}^{t-} e^{-\delta s} H'(U_{s-}^{\pi^*}) dG_s^{\pi^*,x} = \beta E_x \int_{0-}^{t-} e^{-\delta s} dG_s^{\pi^*,x}. \end{aligned}$$

Then, it follows that

$$E_x[e^{-\delta t} H(U_{t-}^{\pi^*})] = H(x) - \alpha E_x \int_{0-}^{t-} e^{-\delta s} dL_s^{\pi^*,x} + \beta E_x \int_{0-}^{t-} e^{-\delta s} dG_s^{\pi^*,x}. \quad (4.9)$$

Since $\lim_{t \rightarrow \infty} E_x[e^{-\delta t} H(U_{t-}^{\pi^*})] \leq \lim_{t \rightarrow \infty} E_x[e^{-\delta t} H(B^*)] = 0$, letting $t \rightarrow \infty$ in (4.9) and using the monotone convergence theorem yield

$$H(x) = \alpha E_x \int_{0-}^{\infty} e^{-\delta s} dL_s^{\pi^*,x} - \beta E_x \int_{0-}^{\infty} e^{-\delta s} dG_s^{\pi^*,x} = V(x; \pi^*).$$

□

Lemma 4.3. $V(x; \pi^*)$ is a concave increasing function on $(0, \infty)$.

Proof. Similar to the proof of Lemma 3.3, we use the arguments of Kulenko and Schmidli [20]. Let $x > 0$, $y > 0$, and $l \in (0, 1)$. Consider the strategies $(L^{\pi^*,x}, G^{\pi^*,x})$ and $(L^{\pi^*,y}, G^{\pi^*,y})$ for the initial capitals x and y . Define $L_t = lL_t^{\pi^*,x} + (1-l)L_t^{\pi^*,y}$ and

$G_t = lG_t^{\pi^*,x} + (1-l)G_t^{\pi^*,y}$. Then, $L_t = L_t^{\pi^*,lx+(1-l)y}$. So, we have

$$\begin{aligned}
& lx + (1-l)y + \int_0^t \mathcal{E}(R)_{s-}^{-1} dP_s - \rho\sigma_p\sigma_R \int_0^t \mathcal{E}(R)_{s-}^{-1} ds \\
& - \int_0^t \mathcal{E}(R)_{s-}^{-1} (l dL_s^{\pi^*,x} + (1-l) dL_s^{\pi^*,y}) \\
& + \int_0^t \mathcal{E}(R)_{s-}^{-1} (l dG_s^{\pi^*,x} + (1-l) dG_s^{\pi^*,y}) \\
& = l \left\{ x + \int_0^t \mathcal{E}(R)_{s-}^{-1} dP_s - \rho\sigma_p\sigma_R \int_0^t \mathcal{E}(R)_{s-}^{-1} ds \right. \\
& \quad \left. - \int_0^t \mathcal{E}(R)_{s-}^{-1} dL_s^{\pi^*,x} + \mathcal{E}(R)_t \int_0^t \mathcal{E}(R)_{s-}^{-1} dG_s^{\pi^*,x} \right\} \\
& + (1-l) \left\{ y + \int_0^t \mathcal{E}(R)_{s-}^{-1} dP_s - \rho\sigma_p\sigma_R \int_0^t \mathcal{E}(R)_{s-}^{-1} ds \right. \\
& \quad \left. - \int_0^t \mathcal{E}(R)_{s-}^{-1} dL_s^{\pi^*,y} + \mathcal{E}(R)_t \int_0^t \mathcal{E}(R)_{s-}^{-1} dG_s^{\pi^*,y} \right\} \geq 0.
\end{aligned}$$

This shows that the strategy (L_t, G_t) is admissible and that

$$G_t^{\pi^*,lx+(1-l)y} \leq lG_t^{\pi^*,x} + (1-l)G_t^{\pi^*,y}.$$

It follows that

$$\begin{aligned}
V(lx + (1-l)y, \pi^*) &= E \left(\alpha \int_{0-}^{\infty} e^{-\delta t} dL_t^{\pi^*,lx+(1-l)y} - \beta \int_{0-}^{\infty} e^{-\delta t} dG_t^{\pi^*,lx+(1-l)y} \right) \\
&\geq lE \left(\alpha \int_{0-}^{\infty} e^{-\delta t} dL_t^{\pi^*,x} - \beta \int_{0-}^{\infty} e^{-\delta t} dG_t^{\pi^*,x} \right) \\
&\quad + (1-l)E \left(\alpha \int_{0-}^{\infty} e^{-\delta t} dL_t^{\pi^*,y} - \beta \int_{0-}^{\infty} e^{-\delta t} dG_t^{\pi^*,y} \right) \\
&= lV(x, \pi^*) + (1-l)V(y, \pi^*),
\end{aligned}$$

which implies the concavity of V . The proof of increasingness of $V(x; \pi^*)$ is routine. \square

4.3 Verification of optimality

Define the barrier level as

$$B^* = \sup\{B \geq 0 : H'(B-) = \alpha\}.$$

We conjecture that the barrier strategy π^* is optimal.

Theorem 4.1. *The value function H defined in (4.7) satisfies*

$$H(x) = V_c(x) = \sup_{\xi_c \in \Xi_c} V_{\xi_c}(x),$$

and the joint strategy $\pi^ = (L^{\pi^*}, G^{\pi^*})$ is optimal, where (L^{π^*}, G^{π^*}) is given by (4.5) and (4.6).*

Proof. Note that $(\mathcal{L} - \delta)H(x) = 0$ and $\alpha \leq H'(x) \leq \beta$ for $x \in [0, B^*)$ due to the concavity of H on $[0, B^*)$. For $x \geq B^*$ and $H(x) = \alpha(x - B^*) + H(B^*)$, we have

$$\begin{aligned} (\mathcal{L} - \delta)H(x) &= -p\alpha + \alpha \int_1^\infty y\Pi(dy) - \alpha(x - B^*) - \delta H(B^*) \\ &< -p\alpha + \alpha \int_1^\infty y\Pi(dy) - \delta H(B^*) \\ &= \lim_{x \rightarrow B^*+} (\mathcal{L} - \delta)H(x) = \lim_{x \rightarrow B^*-} (\mathcal{L} - \delta)H(x) = 0. \end{aligned}$$

Due to the continuity of H, H' and H'' at $x = B^*$. Thus, the function H satisfies the HJB equation (4.2). By Lemma 4.1, we get $H(x) \geq V_c(x)$. On the other hand, $H(x) \leq V_c(x)$. Thus, $H(x) = V_c(x)$. \square

4.4 Two closed-form solutions

We now present two examples in which closed-form solution can be derived.

Example 4.1. Assume that $r = 0$ and $\sigma_R = 0$. Then, $H(x)$ satisfies the following integro-differential equation

$$\mathcal{A}H(x) = \delta H(x), \quad 0 < x < B^*, \quad (4.10)$$

and

$$H(x) = \alpha(x - B^*) + H(B^*), \quad x > B^*, \quad (4.11)$$

with the boundary conditions

$$H'(0) = \beta, \quad H'(B^*) = \alpha, \quad (4.12)$$

where

$$\mathcal{A}g(x) = \frac{1}{2}\sigma_p^2 g''(x) - pg'(x) - \lambda g(x) + \lambda \int_0^\infty g(x+y)F(dy).$$

Again, using the arguments of Laplace transform, one can show that the solution to (4.10) and (4.11) is given by

$$H(x) = -\alpha \bar{Z}^{(\delta)}(B^* - x) + \alpha \frac{E[X_1]}{\delta},$$

and

$$B^* = (Z^{(\delta)})^{-1} \left(\frac{\beta}{\alpha} \right),$$

where $Z^{(\delta)}(x)$ and $\bar{Z}^{(\delta)}(x)$ are defined in Example 3.1. In the case of $\alpha = 1$, these formulae were obtained in Bayraktar, Kyprianou and Yamazaki [11] by using the fluctuation theory of spectrally positive Lévy processes.

Example 4.2. Let $\sigma_R = \sigma_p = 0$. Assume that X_i is exponentially distributed with parameter μ . Then, by Theorem 4.1 and Lemma 4.2, $H(x)$ satisfies the following integro-differential equation

$$(rx - p)H'(x) + \lambda\mu \int_0^\infty H(x+z)e^{-\mu z}dz = (\lambda + \delta)H(x), \quad 0 < x < B^*, \quad (4.13)$$

and

$$H(x) = \alpha(x - B^*) + H(B^*), \quad x > B^*, \quad (4.14)$$

with the boundary conditions

$$H'(0) = \beta, \quad H'(B^*) = \alpha. \quad (4.15)$$

Repeating the steps in Example 3.2, we obtain

$$H(x) = C_3 M \left(-\frac{\delta}{r}, 1 - \frac{\lambda + \delta}{r}, \mu \left(x - \frac{p}{r} \right) \right) + C_4 U \left(-\frac{\delta}{r}, 1 - \frac{\lambda + \delta}{r}, \mu \left(x - \frac{p}{r} \right) \right).$$

The constants C_3 and C_4 can be determined from the boundary conditions (4.15). Using the formulae

$$M'(a, b, z) = \frac{a}{b} M(a+1, b+1, z), \quad U'(a, b, z) = -a U(a+1, b+1, z),$$

we get

$$C_3 = \frac{\beta \Delta_4 - \alpha \Delta_2}{\Delta_1 \Delta_4 - \Delta_2 \Delta_3},$$

and

$$C_4 = \frac{\alpha\Delta_1 - \beta\Delta_3}{\Delta_1\Delta_4 - \Delta_2\Delta_3},$$

where

$$\begin{aligned}\Delta_1 &= -\frac{\mu\delta}{r-\lambda-\delta}M\left(1-\frac{\delta}{r}, 2-\frac{\lambda+\delta}{r}, -\frac{\mu p}{r}\right), \\ \Delta_2 &= \frac{\mu\delta}{r}U\left(1-\frac{\delta}{r}, 2-\frac{\lambda+\delta}{r}, -\frac{\mu p}{r}\right), \\ \Delta_3 &= -\frac{\mu\delta}{r-\lambda-\delta}M\left(1-\frac{\delta}{r}, 2-\frac{\lambda+\delta}{r}, \mu(B^* - \frac{p}{r})\right), \\ \Delta_4 &= \frac{\mu\delta}{r}U\left(1-\frac{\delta}{r}, 2-\frac{\lambda+\delta}{r}, \mu(B^* - \frac{p}{r})\right).\end{aligned}$$

Here, B^* is the unique solution to the following equation with respect to b :

$$-\frac{\mu\delta}{r-\lambda-\delta}C_3M\left(1-\frac{\delta}{r}, 2-\frac{\lambda+\delta}{r}, \mu(b-\frac{p}{r})\right) + \frac{\mu\delta}{r}U\left(1-\frac{\delta}{r}, 2-\frac{\lambda+\delta}{r}, \mu(b-\frac{p}{r})\right) = \alpha.$$

5 Solution to the problem without constraints

We now consider the control problem (2.6) without any restrictions on capital injections.

In this case, ruin can occur and the time of ruin for a control strategy ξ is defined as

$$\tau_\xi = \inf\{t : U_t^\xi = 0\},$$

because of the diffusion and the skip-free downward surplus process. Then, it follows from (3.1), (4.1) and (2.5) that for all $x \geq 0$, $V_\xi(x) \geq \max\{V_d(x), V_c(x)\}$. We shall determine V_* and the optimal strategy ξ^* such that $V_*(x) = V(x; \xi^*)$.

5.1 Verification lemma

For the control problem without any restrictions on capital injections, we get the following associated HJB equation:

$$\max\{\mathcal{L}v(x) - \delta v(x), \quad \alpha - v'(x), v'(x) - \beta\} = 0, \quad x \geq 0, \quad (5.1)$$

with the boundary condition

$$\max\{-v(0), v'(0) - \beta\} = 0. \quad (5.2)$$

Lemma 5.1. (*Verification Lemma*) If v satisfies the HJB equation (5.1) with the boundary condition (5.2), then $v(x) \geq V_\xi(x)$ for any admissible policy ξ .

Proof. For any admissible strategy $\xi \in \Xi$, put $\Lambda = \{s : L_{s-}^\xi \neq L_s^\xi\}$. Applying Ito's formula for semimartingale to $e^{-\delta t}v(U_t^\xi)$ gives

$$\begin{aligned} E_x[e^{-\delta(t \wedge \tau_\xi)}v(U_{t \wedge \tau_\xi-}^\xi)] &= v(x) + E_x \int_0^{t \wedge \tau_\xi-} e^{-\delta s}(\mathcal{L} - \delta)v(U_{s-}^\xi)ds \\ &\quad + E_x \sum_{s \in \Lambda, s \leq t \wedge \tau_\xi-} e^{-\delta s} \left\{ v(U_s^\xi) - v(U_{s-}^\xi) \right\} \\ &\quad - E_x \int_{0-}^{t \wedge \tau_\xi-} e^{-\delta s} v'(U_{s-}^\xi) dL_s^{\xi, c} \\ &\quad + E_x \int_{0-}^{t \wedge \tau_\xi-} e^{-\delta s} v'(U_{s-}^\xi) dG_s^\xi, \end{aligned} \quad (5.3)$$

where $L_s^{\xi, c}$ is the continuous part of L_s^ξ . We see from (5.1) that $(\mathcal{L} - \delta)v(U_{s-}^\xi) \leq 0$ and $\alpha \leq v'(x) \leq \beta$. Thus,

$$E_x \int_{0-}^{t \wedge \tau_\xi-} e^{-\delta s} v'(U_{s-}^\xi) dG_s^\xi \leq \beta E_x \int_{0-}^{t \wedge \tau_\xi-} e^{-\delta s} dG_s^\xi, \quad (5.4)$$

and for $s \in \Lambda, s \leq t \wedge \tau_\xi$,

$$v(U_s^\xi) - v(U_{s-}^\xi) \leq -\alpha(L_s^\xi - L_{s-}^\xi). \quad (5.5)$$

It follows from (5.3) and (5.5) that

$$E_x[e^{-\delta(t \wedge \tau_\xi)}v(U_{t \wedge \tau_\xi-}^\xi)] \leq v(x) - \alpha E_x \int_{0-}^{t \wedge \tau_\xi-} e^{-\delta s} dL_s^\xi + \beta E_x \int_{0-}^{t \wedge \tau_\xi-} e^{-\delta s} dG_s^\xi. \quad (5.6)$$

Finally, by letting $t \rightarrow \infty$ in (5.6) and noting that (by Fatou's lemma)

$$\liminf_{t \rightarrow \infty} E_x[e^{-\delta(t \wedge \tau_\xi)}v(U_{t \wedge \tau_\xi-}^\xi)] \geq E_x[\liminf_{t \rightarrow \infty} e^{-\delta(t \wedge \tau_\xi)}v(U_{t \wedge \tau_\xi}^\xi)] \geq v(0)E_x[e^{-\delta\tau_\xi}] \geq 0,$$

we prove the lemma. □

5.2 Construction of a candidate solution

For any $x \geq 0$, we set our candidate strategy to be

$$\xi^* = \begin{cases} \xi_d^*, & \text{if } V_{b^*}'(0) \leq \beta, \\ \xi_c^*, & \text{if } H(0) \geq 0, \end{cases} \quad (5.7)$$

and our candidate solution to be

$$V_{\xi^*}(x) = \begin{cases} V_d(x), & \text{if } V_{b^*}'(0) \leq \beta, \\ V_c(x), & \text{if } H(0) \geq 0, \end{cases} \quad (5.8)$$

where V_d and V_c are given by (3.1) and (4.1), respectively, and V_{b^*} and H are given by (3.6) and (4.7), respectively.

5.3 Verification of optimality

Theorem 5.1. *The value function V_{ξ^*} defined in (5.8) satisfies*

$$V_{\xi^*}(x) = V_*(x) = \sup_{\xi \in \Xi} V(x; \xi),$$

and the joint strategy ξ^ defined in (5.7) is optimal.*

Proof. If $V_{b^*}'(0) \leq \beta$, then V_{b^*} satisfies the equation (5.1) with the condition (5.2). Hence, $V_{b^*}(x) \geq V_*(x)$. On the other hand, $V_{b^*}(x) = V(x; \xi_d^*) \leq V_d(x)$. It follows that $V_{\xi^*}(x) = V_{b^*}(x) = V_d(x)$. The optimality of ξ_d^* is verified by Theorem 3.1. If $H(0) \geq 0$, then H satisfies the HJB equation (4.1), so that $H(x) \leq V_c(x)$. Since H also satisfies the equation (5.1) with the condition (5.2), $H(x) \geq V(x; \xi_c^*) \geq V_c(x)$. Hence, we have $V_{\xi^*}(x) = V(x; \xi_c^*) = V_c(x)$. The optimality of ξ_c^* is verified by Theorem 4.1. \square

Acknowledgements

The authors would like to thank two anonymous referees and the editor for their helpful comments on the previous version of the paper. The research of Chuancun Yin was supported by the National Natural Science Foundation of China (No. 11171179) and the Research Fund for the Doctoral Program of Higher Education of China (No. 20133705110002). The research of Kam C. Yuen was supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. HKU 7057/13P).

References

- [1] M. Abramowitz and A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Publications, New York, 1965.
- [2] S. Asmussen, F. Avram and M. R. Pistorius, Russian and American put options under exponential phase-type Lévy models, *Stochastic Processes and their Applications*, **109** (2004), 79-111.
- [3] B. Avanzi, Strategies for dividend distribution: A review, *North American Actuarial Journal*, **13** (2009), 217-251.
- [4] B. Avanzi, E. C. K. Cheung, B. Wong and J.-K. Woo, On a periodic dividend barrier strategy in the dual model with continuous monitoring of solvency, *Insurance: Mathematics and Economics*, **52** (2013), 98-113.
- [5] B. Avanzi and H. U. Gerber, Optimal dividends in the dual model with diffusion, *ASTIN Bulletin*, **38** (2008), 653-667.
- [6] B. Avanzi, H. U. Gerber and E. S. W. Shiu, Optimal dividends in the dual model, *Insurance: Mathematics and Economics*, **41** (2007), 111-123.
- [7] B. Avanzi, J. Shen and B. Wong, Optimal dividends and capital injections in the dual model with diffusion, *ASTIN Bulletin*, **41** (2011), 611-644.
- [8] B. Avanzi, V. Tu and B. Wong, On optimal periodic dividend strategies in the dual model with diffusion, *Insurance: Mathematics and Economics*, **55** (2014), 210-224.
- [9] P. Azcue and N. Muler, Optimal reinsurance and dividend distribution policies in the Cramér-Lundberg model, *Mathematical Finance*, **15** (2005), 261-308.
- [10] E. Bayraktar and M. Egami, Optimizing venture capital investments in a jump diffusion model, *Mathematical Methods of Operations Research*, **67** (2008), 21-42.
- [11] E. Bayraktar, A. E. Kyprianou and K. Yamazaki, On optimal dividends in the dual model, *ASTIN Bulletin*, **43** (2013), 359-372.

- [12] E. Bayraktar, A. E. Kyprianou and K. Yamazaki, Optimal dividends in the dual model under transaction costs, *Insurance: Mathematics and Economics*, **54** (2014), 133-143.
- [13] E. C. K. Cheung and S. Dreikic, Dividend moments in the dual model: Exact and approximate approaches, *ASTIN Bulletin*, **38** (2008), 149-159.
- [14] H. Dai, Z. Liu and N. Luan, Optimal dividend strategies in a dual model with capital injections, *Mathematical Methods of Operations Research*, **72** (2010), 129-143.
- [15] H. Dai, Z. Liu and N. Luan, Optimal financing and dividend control in the dual model, *Mathematical and Computer Modelling*, **53** (2011), 1921-1928.
- [16] B. De Finetti, Su un'impostazione alternativa della teoria collettiva del rischio, *Transactions of the XVth International Congress of Actuaries*, **2** (1957), 433-443.
- [17] W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*, Applications of Mathematics, Springer-Verlag, New York, 1993.
- [18] L. He and Z. Liang, Optimal financing and dividend control of the insurance company with fixed and proportional transaction costs, *Insurance: Mathematics and Economics*, **44** (2009), 88-94.
- [19] S. Jaschke, A note on the inhomogeneous linear stochastic differential equation, *Insurance: Mathematics and Economics*, **32** (2003), 461-464.
- [20] N. Kulenko and H. Schmidli, Optimal dividend strategies in a Cramér-Lundberg model with capital injections, *Insurance: Mathematics and Economics*, **43** (2008), 270-278.
- [21] K. Miyasawa, An economic survival game, *Journal of the Operations Research Society of Japan*, **4** (1962), 95-113.
- [22] H. Schmidli, *Stochastic Control in Insurance*, Springer, New York, 2008.

- [23] D. J. Yao, H. L. Yang and R. M. Wang, Optimal financing and dividend strategies in a dual model with proportional costs, *Journal of Industrial and Management Optimization*, **6** (2010), 761-777.
- [24] D. J. Yao, H. L. Yang and R. W. Wang, Optimal dividend and capital injection problem in the dual model with proportional and fixed transaction costs, *European Journal of Operational Research*, **211** (2011), 568-576.
- [25] D. J. Yao, R. W. Wang and L. Xu, Optimal dividend and capital injection strategy with fixed costs and restricted dividend rate for a dual model, *Journal of Industrial and Management Optimization*, **10** (2014), 1235-1259.
- [26] C. C. Yin and Y. Z. Wen, Optimal dividends problem with a terminal value for spectrally positive Lévy processes, *Insurance: Mathematics and Economics*, **53** (2013), 769-773.
- [27] C. C. Yin and Y. Z. Wen, An extension of Paulsen-Gjessing's risk model with stochastic return on investments, *Insurance: Mathematics and Economics*, **52** (2013), 469-472.
- [28] C. C. Yin, Y. Z. Wen and Y. X. Zhao, On the optimal dividend problem for a spectrally positive Lévy process, *ASTIN Bulletin*, **44** (2014), 635-651.
- [29] Z. M. Zhang, On a risk model with randomized dividend-decision times, *Journal of Industrial and Management Optimization*, **10** (2014), 1041-1058.